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ON THE OCCURRENCE OF DOUBLY PERIODIC CONVECTION IN A HORIZONTAL LAYER

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We study the stationary, doubly periodic convective modes in a horizontal fluid layer, in which the loss of stability is accompanied by bifurcation from the state of rest. The fluid layer is heated from below.

The bifurcation equation is four-dimensional for almost every ratio of the wave numbers. Using the fact that the problem is invariant with respect to the shears in the horizontal plane, we succeed in reducing the dimensionality of the bifurcation equation. Study of the resulting two-dimensional system shows that the state of rest gives rise to three, substantially different modes. One of these modes represents a rectangular convection [1, 2] and the remaining two, a plane convection. The assertions made in the present work are also valid in the case of convection in a layer with two free boundaries.

When determining the secondary, doubly periodic convective modes, we usually make certain additional assumptions about their symmetry. This makes it possible to reduce the problem to the case of a simple spectrum [1 – 3]. However, the problem of determining all doubly periodic modes requires that the complete multi-dimensional bifurcation equation be considered. Such a problem is posed in e. g. [4].

The group theoretic properties of the problem are found helpful in investigating the bifurcation equation. Thus in [5] it is shown that in certain cases (in particular in the case of a plane convection) the group theoretic considerations make it possible to reduce the investigation of bifurcation in the presence of a multiple spectrum, to a one-dimensional bifurcation equation. In the case of rectangular convection however, the method of [5] succeeds only in reducing the order of the system of bifurcation equations to two.

The purpose of this paper is to study all possible stationary modes of doubly periodic free convection in a layer, occurring when the Rayleigh number passes through its minimum critical value.

1. Statement of the problem. Let us consider the convection in a fluid layer bounded by two immovable horizontal planes $z = h$ and $z = -h$ maintained at a constant temperature

$$\nu\Delta\mathbf{v} - \frac{1}{\rho}\nabla p = (\mathbf{v}\nabla)\mathbf{v} + \beta\mathbf{g}T, \quad \chi\Delta T = \mathbf{v}\nabla T, \quad \operatorname{div}\mathbf{v} = 0$$

$$T(x, y, h) = T_1, \qquad T(x, y, -h) = T_2, \qquad \mathbf{v}(x, y, \pm h) = 0 \tag{1.1}$$

Here x, y, z are rectangular Cartesian coordinates with the z-axis pointing vertically downwards; ρ, ν, χ and β are the constants characterizing the fluid, and g is acceleration due to gravity. The problem (1.1) has the following stationary solution corresponding to the state of rest

$$v_0 = 0,\ T_0 = Cz + C_1,\ p_0 = \rho\beta g\ (Cz^2/2 + C_1 z + C_2)$$

We seek another stationaty solution v', p' and T' in the form

$$v' = v_0 + \frac{\nu}{2h} V, \qquad p' = p_0 + \frac{\nu^2 \rho}{(2h)^2} q, \qquad T' = T_0 + \frac{2h\nu C}{\chi \sqrt{R}} \tau$$

$$\left(R = \frac{\beta g C\,(2h)^4}{\nu\chi}\right)$$

We obtain the following system in the dimensionless variables (the notation of coordinates is retained) for the perturbations:

$$\Delta V - \nabla q = (V\nabla)\, V + \gamma\tau k, \qquad \Delta\tau = PV\nabla\tau + \gamma V_z \tag{1.2}$$

$$\operatorname{div} V = 0, \quad \gamma = \sqrt{R}, \quad P = \nu / \chi, \quad k = (0, 0, 1)$$

$$V = 0, \qquad \tau = 0, \qquad z = \pm {}^1/_2 \tag{1.3}$$

We shall be interested in those solutions of the problem (1.2), (1.3) which are $2\pi/\alpha$-periodic in x and $2\pi/\beta$-periodic in y. Let in addition assume that the layer, as a whole, undergoes no displacement in the horizontal plane

$$\int_{-1/2}^{1/2} \int_{-\pi/\beta}^{\pi/\beta} V_x dy\, dz = \int_{-1/2}^{1/2} \int_{-\pi/\alpha}^{\pi/\alpha} V_y dx\, dz = 0 \tag{1.4}$$

Let us consider the following Hilbert spaces:

1) Space H^1 which is the closure of smooth periodic vectors V defined in the layer and satisfying the condition (1.4) in the metric

$$(V_1, V_2)_{H^1} = \frac{\alpha\beta}{4\pi^2} \int_\Omega V_1 V_2 d\Omega$$

Here Ω denotes the parallelepiped $|x| \leqslant \pi/\alpha,\ |y| \leqslant \pi/\beta,\ |z| \leqslant {}^1/_2$.

2) Space H^2 which is the closure of a set of smooth periodic functions τ defined in the layer and vanishing at the layer boundary in the metric

$$(\tau_1, \tau_2)_{H^2} = \frac{\alpha\beta}{4\pi^2} \int_\Omega \tau_1 \tau_2 d\Omega$$

3) Subspace H_0^1 of H^1 representing the closure on the norm of H^1 of a set of smooth solenoidal vectors $V \in H^1$ vanishing at the layer boundary.

4) Space $H = H^1 \times H^2$.

5) Space $H_0 = H_0^1 \times H^2$.

In the following we shall denote the elements of H by $\Phi = \{V, \tau\} = \{V_x, V_y, V_z, \tau\}$, $V \in H^1$, $\tau \in H^2$.

Let Π be the orthogonal projection operator from H into H_0 and let M be a set of twice continuously differentiable vectors $\Phi \in H_0$ satisfying the conditions (1.3). The following operators act from M into H_0:

$$\begin{gathered} A\Phi = -\Pi\,\{\Delta V, \Delta\tau\}, \qquad B\Phi = -\Pi\,\{\tau k, V_z\} \\ K\,[\Phi_1, \Phi_2] = -\Pi\,\{(V_1\nabla)\, V_2, PV_1\nabla\tau_2\}, \quad K\Phi = K\,[\Phi, \Phi] \\ K^\circ\,[\Phi_1, \Phi_2] = K\,[\Phi_1, \Phi_2] + K\,[\Phi_2, \Phi_1] \end{gathered} \tag{1.5}$$

In the new notation the system (1.2) – (1.4) assumes the form

$$A\Phi = \gamma B\Phi + K\Phi \tag{1.6}$$

The operator B is symmetrical and A is positive definite. Equation (1.6) yields

$$\Phi = \gamma A^{-1}B\Phi + A^{-1}K\Phi \equiv F_\gamma \Phi$$

The operator F_γ is completely continuous in the energetic space H_A of the operator A.

Let $\gamma = \gamma_0$ denote the point of bifurcation of Eq. (1.6). We consider, for small $\mu > 0$, the following problem

$$(A - \gamma_0 B)\Phi = \mu B\Phi + K\Phi \tag{1.7}$$

The study of bifurcation is made more difficult here by the fact that the problem in question has a great number of solutions. We shall indicate one of the reasons for this. Let us determine, in the space H, the shear operator in x, y

$$L_{\xi\eta}\Phi = \Phi_1, \quad \Phi_1\ (x, y, z) = \Phi(x + \xi,\ y + \eta,\ z)$$

The operators A, B and K are invariant with respect to the shears

$$L_{\xi\eta}A = AL_{\xi\eta}, \qquad L_{\xi\eta}B = BL_{\xi\eta}, \qquad L_{\xi\eta}K[\Phi_1, \Phi_2] = K[L_{\xi\eta}\Phi_1, L_{\xi\eta}\Phi_2] \tag{1.8}$$

We note that the operator $L_{\xi\eta}$ is isometric

$$(L_{\xi\eta}\Phi_1, L_{\xi\eta}\Phi_2)_H = (\Phi_1, \Phi_2)_H$$

Let Φ be any solution of Eq. (1.7). Then by virtue of the relations (1.8) the vectors $L_{\xi\eta}\Phi$ will also be its solution for any ξ, η. Thus, in the present case the invariance of the problem with respect to the horizontal shears defines, in many instances, the multiplicity of the branches.

Let us separate the set of all solutions of (1.7) into classes, collecting in each class solutions generated from each other by shears. This can be done as the shear transformation has an inverse. Each solution belongs to one and only one class. We shall call a set of solutions containing one element from each class a complete set. Our problem consists of finding a complete set of small solutions of the problem (1.7).

2. The linearised problem. Let us consider the following linear problem:

$$A\Phi = \gamma_0 B\Phi \tag{2.1}$$

By definition (2.1) is equivalent to the system

$$\Delta \mathbf{V} - \nabla q = \gamma_0 \tau \mathbf{k}, \qquad V_z = \frac{1}{\gamma_0}\Delta\tau, \qquad \{\mathbf{V}, \tau\} \in M \tag{2.2}$$

Eliminating $\mathbf{V}$ and q, we obtain from (2.2)

$$\Delta^3\tau = \gamma_0^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\tau, \quad \tau = \Delta\tau = \frac{\partial\Delta\tau}{\partial z} = 0, \quad z = \pm {}^1/_2 \tag{2.3}$$

Let H° and H_τ° denote the solution spaces of (2.1) and (2.3), respectively, and consider the vector $\{\mathbf{V}, \tau\} = \Phi \in H^\circ$. With τ specified, $\mathbf{V}$ can be uniquely determined from (2.2), therefore

$$\Phi = N\tau, \ \Phi \in H^\circ$$

By virtue of the last relation the spaces H° and H_τ° are isomorphic. If $\{\tau_i^\circ\}_1^n$ is a basis in H_τ°, then $\{N\tau_i^\circ\}_1^n$ is a basis in H°. Seeking the solution of the problem (2.3) in the form $\tau = u(z)e^{i(\alpha l x + \beta m y)}$ we arrive at the following equation:

$$-L^3u = \lambda u, \quad u = Lu = Lu' = 0, \quad z = \pm 1/_2 \qquad (2.4)$$
$$L = d^2/dz^2 - \theta^2, \qquad \lambda = \gamma_0^2\theta^2, \quad \theta^2 = \alpha^2l^2 + \beta^2m^2$$

As we know [1], the problem (2.4) has simple eigenvalues. Let $\lambda_0(\theta)$ be the first eigenvalue of the problem (2.4). We assume

$$\gamma_0^2 = \min_\theta \frac{\lambda_0(\theta)}{\theta^2} = \frac{\lambda_0(\theta_0)}{\theta_0^2} \qquad (2.5)$$

The value $\gamma = \gamma_0$ represents the point of bifurcation of the problem (1.6) [1]. (The uniqueness of the extremum (2.5) is not proved strictly, however numerous computations indicate that this is, in fact, the case).

Let us consider the set E of all pairs (α, β) such that the relation

$$\alpha^2l^2 + \beta^2m^2 = \theta_0^2 \qquad (2.6)$$

holds for one and only one pair (l, m) of natural numbers. For $(\alpha, \beta) \in E$ the space H_τ° is four-dimensional. The following functions can be chosen as the basis functions:

$$\tau_1^\circ = u(z)\cos(\alpha lx + \beta my), \quad \tau_2^\circ = u(z)\cos(\alpha lx - \beta my)$$
$$\tau_3^\circ = u(z)\sin(\alpha lx + \beta my), \quad \tau_4^\circ = u(z)\sin(\alpha lx - \beta my) \qquad (2.7)$$

We shall use the vectors $\Phi_i^\circ = N\tau_i^\circ$, $i = 1, \ldots, 4$ as the basis in H°. By (1.8) the space H° is invariant under the transformations $L_{\xi\eta}$. If $\Phi^\circ \in H^\circ$, then $L_{\xi\eta}\Phi^\circ \in H^\circ$ also. As before, we divide H° into classes, collecting in each class all the vectors generated from each other by shears in x, y. The following lemma establishes the one-to-one correspondence between each such class belonging to H°, and a unique vector of specific form belonging to the same class.

Lemma 1. For any nonzero vector $\Phi^\circ \in H^\circ$ we can find ξ and η such, that

$$L_{\xi\eta}\Phi^\circ = C_1^\circ\Phi_1^\circ + C_2^\circ\Phi_2^\circ, \qquad C_i^\circ \geqslant 0 \qquad (2.8)$$

where C_i° are uniquely defined. (All lemmas are given without proof because of the limit imposed on the length of this paper).

We shall note another property of (2.1) which will be found important. The inequality

$$(A\Phi - \gamma_0 B\Phi, \ \Phi)_H \geqslant 0 \qquad (2.9)$$

holds for all $\Phi \in M$ and equality in (2.9) is possible only when $\Phi \in H^\circ$ (see e.g. [3]).

3. Mirror reflections.

Let us use the following notation:

$$\{V_x, V_y, V_z\}(\xi, \eta, \zeta) = \{V_x(\xi, \eta, \zeta), V_y(\xi, \eta, \zeta), V_z(\xi, \eta, \zeta)\}$$

We define the "mirror reflection" operators S_1 and S_2 for the vectors $\mathbf{V}$ and Φ and for the functions of τ as follows:

$$S_1\mathbf{V} = \{-V_x, V_y, V_z\}(-x, y, z), \qquad S_2\mathbf{V} = \{V_x, -V_y, V_z\}(x, -y, z)$$
$$S_1\tau = \tau(-x, y, z), \quad S_2\tau = \tau(x, -y, z), \quad S_i\Phi = \{S_i\mathbf{V}, S_i\tau\}$$

By direct verification we can establish the following properties of the operators S_i:

$$S_iA = AS_i, \quad S_iB = BS_i, \quad S_iK[\Phi_1, \Phi_2] = K[S_i\Phi_1, S_i\Phi_2] \qquad (3.1)$$

The operators S_i are isometric

$$(S_i\Phi_1, S_i\Phi_2)_H = (\Phi_1, \Phi_2)_H \qquad (3.2)$$

We denote $S = S_1S_2$. Let H^c be a set of vectors $\Phi_1 \in H_0$, $S\Phi_1 = \Phi$, and H^s as a set of vectors $\Phi_2 \in H_0$, $S\Phi_2 = -\Phi$. By virtue of the relation (3.2), the above sets are orthogonal subspaces

$$(\Phi_1, \Phi_2)_H = 0, \ \Phi_1 \in H^c, \ \Phi_2 \in H^s \tag{3.3}$$

We note that out of the previously selected basis vectors,

$$\Phi_1^\circ, \Phi_2^\circ \in H^c, \quad \Phi_3^\circ, \Phi_4^\circ \in H^s$$

4. Complete set of solutions. We seek a solution of the problem (1.7) in the form

$$\Phi = \Phi^\circ + \Phi_0, \ \Phi^\circ \in H^\circ, \ \Phi_0 \perp H^\circ \tag{4.1}$$

Let Q be the projector on the orthogonal complement of the subspace H°, and $A - \gamma_0 B = D$. The problem (1.7) can be written in the following equivalent form

$$D\Phi_0 = Q\,[\mu B\,(\Phi^\circ + \Phi_0) + K\,(\Phi^\circ + \Phi_0)], \quad \Phi_0 \perp H^\circ \tag{4.2}$$

$$(\mu B\Phi + K\Phi, \ \Phi_k^\circ)_H = 0, \qquad k = 1, \ldots, 4 \tag{4.3}$$

For given small μ and Φ° Eq. (4.2) has a unique small solution $\Phi_0 = \Phi_0\,(\mu, \Phi^\circ)$.

Lemma 2. For any solution Φ of the problem (4.1) – (4.3), ξ and η can be found such, that $L_{\xi\eta}\,\Phi \in H^c$.

This lemma follows from the previous remark, Lemma 1 and the relations (3.1)–(3.3). Lemma 2 enables us to reduce the problem of obtaining a complete set of small nonzero solutions of (1.7) to that of finding a complete set of solutions belonging to H^c.

In the following we shall find useful the relations obtained by integration by parts:

$$(K\,[\Phi_1, \Phi_2], \Phi_3)_H = -\,(K\,[\Phi_1, \Phi_3], \Phi_2)_H, \ \Phi_i \in M \tag{4.4}$$

$$(K\,[\Phi_1, \Phi_2], \Phi_2)_H = 0 \tag{4.5}$$

We note another important property of the problem (1.7), namely that for $\mu = 0$ it has no nonzero solutions [5].

Lemma 3. The vector $K\,[\Phi_j^\circ, \Phi_k^\circ]$ is orthogonal to the space H°.

Let Φ_{jk} be a solution of the following problem:

$$D\Phi_{jk} = K\,[\Phi_j^\circ, \Phi_k^\circ], \qquad \Phi_{jk} \perp H^\circ \tag{4.6}$$

The above problem has, by virtue of Lemma 3, a unique solution.

Theorem 1. For all $(\alpha, \beta) \in E$ and the Prandtl numbers such that

$$I = (D\Phi_{11}, \ \Phi_{11})_H - (D\Phi_{11}, \ \Phi_{22})_H - (D\Phi_{12}, \ \Phi_{12} + \Phi_{21})_H \neq 0 \tag{4.7}$$

nonzero solutions of the problem (1.7) can be written, for small μ, in the form

$$\Phi = \sum_{k=1}^{\infty} \Phi_k \varepsilon^k, \qquad \varepsilon = \sqrt{\mu} \tag{4.8}$$

These solutions are either plane flows, or can be obtained by shears from a convective flow with rectangular cell, symmetrical with respect to the XOZ and YOZ planes.

Proof. By virtue of Lemmas 1 and 2, the vector Φ° can be written in the form

$$\Phi^\circ = \alpha_1\Phi_1^\circ + \alpha_2\Phi_2^\circ, \quad \alpha_i \geqslant 0 \tag{4.9}$$

Equation (4.2) yields a unique expression for $\Phi_0 = \Phi_0\,(\alpha_1, \alpha_2, \mu)$. We seek Φ_0 in the form of a series

$$\Phi_0 = \sum_{i,\,j,\,k=0}^{\infty} \Phi_{ijk}\alpha_1{}^i\alpha_2{}^j\mu^k, \qquad \Phi_{ijk} \perp H^\circ, \qquad \Phi_{ijk} \in H^\circ \tag{4.10}$$

where $\Phi_{000} = 0$. Substituting (4.10) into the conditions (4.3) we obtain the following system of bifurcation equations:

$$(\mu B\Phi + K\Phi,\ \Phi_k^\circ)_H \equiv F_k(\alpha_1, \alpha_2, \mu) = 0, \qquad k = 1,2 \tag{4.11}$$

The conditions of solvability for $k = 3,4$ are fulfilled automatically by virtue of the relation (3.3) together with the note which follows it. The number of different solutions of (1.7) belonging to H^c is equal to the number of different solutions $\{\alpha_1(\mu), \alpha_2(\mu)\}$ of (4.11).

Let us establish some properties of the system (4.11) following from the group properties of the initial problem. We set $\mu B\Phi + K\Phi = f(\alpha_1, \alpha_2)$ and note that

$$S_l\Phi_1^\circ = \Phi_2^\circ, \quad S_l\Phi_2^\circ = \Phi_1^\circ \tag{4.12}$$

From (4.12) it follows that $S_l f(\alpha_1, \alpha_2) = f(\alpha_2, \alpha_1)$. Further, taking into account the fact that the operator S_l is isometric, we have

$$(f(\alpha_1, \alpha_2),\ \Phi_2^\circ)_H = (S_l f(\alpha_1, \alpha_2),\ S_l\Phi_2^\circ)_H = (f(\alpha_2, \alpha_1),\ \Phi_1^\circ)_H$$

Thus

$$F_2(\alpha_1, \alpha_2, \mu) = F_1(\alpha_2, \alpha_1, \mu) \tag{4.13}$$

We now set $L_{\pi/2al,\pi/2\beta,m} = L_1$ and note that

$$L_1\Phi_1^\circ = -\Phi_1^\circ, \qquad L_1\Phi_2^\circ = \Phi_2^\circ \tag{4.14}$$

In analogy with the previous argument, (4.14) yield

$$F_2(\alpha_1, \alpha_2, \mu) = F_1(-\alpha_1, \alpha_2, \mu)$$

$$F_1(\alpha_1, \alpha_2, \mu) = -F_1(-\alpha_1, \alpha_2, \mu) \tag{4.15}$$

Using (4.13) and (4.15) we conclude that F_1 contains only the odd powers of α_1 and the even powers of α_2, by interchanging α_1 and α_2, F_2 is obtained from F_1.

Thus system (4.11) has the form

$$b\mu\alpha_1 + a_1\alpha_1^3 + a_2\alpha_1\alpha_2^2 + \ldots = 0 \tag{4.16}$$

$$b\mu\alpha_2 + a_2\alpha_2\alpha_1^2 + a_1\alpha_2^3 + \ldots = 0$$

Let us now determine the coefficients b, a_1 and a_2 Substituting the expression (4.10) into (4.2) and taking into account Lemma 3 and (4.6), we obtain

$$\Phi_{100} = \Phi_{010} = \Phi_{001} = \Phi_{00k} = 0,\ k = 1, 2, \ldots \tag{4.17}$$

$$\Phi_{200} = \Phi_{11},\ \Phi_{020} = \Phi_{22},\ \Phi_{110} = \Phi_{12} + \Phi_{21} \tag{4.18}$$

The coefficients b, a_1 and a_2 are expressed in terms of Φ_i°, Φ_{200}, Φ_{020} and Φ_{110}. Taking into account the relations (4.4), (4.5) and (4.18), we obtain

$$b = (B\Phi_1^\circ,\ \Phi_1^\circ)_H \tag{4.19}$$

$$a_1 = (K^\circ[\Phi_{200}, \Phi_1^\circ],\ \Phi_1^\circ)_H = -(D\Phi_{11},\ \Phi_{11})_H \tag{4.20}$$

Let us consider the expression for a_2

$$a_2 = (K^\circ[\Phi_{110}, \Phi_2^\circ] + K^\circ[\Phi_{020}, \Phi_1^\circ],\ \Phi_1^\circ)_H$$

We note that by virtue of the relations (4.12) and (4.18), $S_l\Phi_{110} = \Phi_{110}$. The latter, together with the identity (4.4) yields

$$(K[\Phi_{110}, \Phi_2^\circ], \Phi_1^\circ)_H = (K[\Phi_{110}, \Phi_1^\circ]_H = 0$$

after which, taking into account (4.4), (4.5) and (4.18) we obtain

$$a_2 = -(D\Phi_{12}),\ \Phi_{12} + \Phi_{21} - (D\Phi_{11},\ \Phi_{22})_H \tag{4.21}$$

We shall show that $b > 0$. In accordance with the definition of the operator B, we have

$$-b = (\Pi\{\tau_1{}^\circ k, V_{1z}{}^\circ\}, \Phi_1{}^\circ)_H = 2(V_{1z}{}^\circ, \tau_1{}^\circ)_{H^1} = \frac{2}{\gamma_0}(\Delta\tau_1{}^\circ, \tau_1{}^\circ)_{H^1} < 0$$

Next we show that $a_1 < 0$. From the relations (4.20) and (4.10) it follows that $a_1 \leqslant 0$ and, that if $a_1 = 0$, then $\Phi_{11} \in H^\circ$. The latter is however impossible, otherwise we would have $K\,\Phi_1{}^\circ = 0$, and the problem (1.7) would have a nonzero solution $\Phi_1{}^\circ$ for $\mu = 0$.

Let us now return to the system (4.16). As the first equation contains the factor α_1 and the second α_2, the system has two obvious solutions

$$\alpha_2 = 0, \qquad \alpha_1{}^2 = \sum_{k=1}^{\infty} t_k \mu^k, \qquad t_1 = -\frac{b}{a_1} > 0$$

$$\alpha_1 = 0, \qquad \alpha_2{}^2 = \sum_{k=1}^{\infty} t_k \mu^k$$

These solutions correspond to plane convective flows. Now assume that $\alpha_1, \alpha_2 \neq 0$. In search of other admissible solutions we divide the equations of (4.16) by α_1 and α_2, respectively. The resulting system contains only even powers of α_1 and α_2

$$b\mu + a_1\alpha_1{}^2 + a_2\alpha_2{}^2 + \ldots = 0 \tag{4.22}$$

$$b\mu + a_2\alpha_1{}^2 + a_1\alpha_2{}^2 + \ldots = 0$$

Assume that the determinant

$$\Delta = \begin{vmatrix} a_1 & a_2 \\ a_2 & a_1 \end{vmatrix} = (a_1 + a_2)(a_1 - a_2)$$

is nonzero. Then the system (4.22) has a unique solution in $\alpha_1{}^2$ and $\alpha_2{}^2$

$$\alpha_1{}^2 = \alpha_2{}^2 = \sum_{k=1}^{\infty} g_k \mu^k, \qquad g_1 = -\frac{b}{a_1 + a_2} > 0$$

(Below we shall show that $a_1 + a_2 < 0$.) In this case the corresponding τ° have the form

$$\tau^\circ = \pm\,\alpha_1 u(z)\,[\cos(\alpha l x + \beta m y) \pm \cos(\alpha l x - \beta m y)]$$

It is obvious that all such solutions are obtained by shears from the "rectangular" convection in which

$$\tau^\circ = 2\alpha_1 u(z)\cos\alpha l x \cos\beta m y$$

It remains to show that the condition (4.7) implies that $\Delta \neq 0$. First we shall show that $a_1 + a_2 < 0$. Taking into account the relations (4.12), (3.4) and (3.5) we obtain from (4.20) and (4.21) the following new expressions for a_1 and a_2

$$a_1 = -(D\Phi_{22},\ \Phi_{22})_H, \quad a_2 = -(D\Phi_{21},\ \Phi_{12} + \Phi_{21})_H - (D\Phi_{22},\ \Phi_{11})_H \tag{4.23}$$

Combining the expressions (4.23), (4.20) and (4.21) we obtain

$$-2(a_1 + a_2) = (D(\Phi_{12} + \Phi_{21}),\ \Phi_{12} + \Phi_{21})_H + (D(\Phi_{11} + \Phi_{22}),\ \Phi_{11} + \Phi_{22})_H \tag{4.24}$$

By virtue of the inequality (2.10) the expression (4.21) is nonnegative and it can become equal to zero only when $\Phi_{11} + \Phi_{22} \in H^\circ$ and $\Phi_{12} + \Phi_{11} \in H^\circ$. This is however impossible, otherwise $K(\Phi_1^\circ + \Phi_2^\circ) = 0$ and the problem (1.7) has a nonzero solution $\Phi = \Phi_1^\circ + \Phi_2^\circ$ with $\mu = 0$. Thus $\Delta \neq 0$ if $a_1 - a_2 \neq 0$ and this leads to the condition (4.7). This completes the proof of Theorem 1.

5. Computation of I. We consider the functional I on the set $(\alpha, \beta) \in E$. Let us denote $\alpha l = \alpha_1$ and $\beta m = \beta_1$. By virtue of the relation (2.7) we have

$$\alpha_1^2 + \beta_1^2 = \theta_0^2 \tag{5.1}$$

The Prandtl number P appears linearly in the operator K. The functional I is of course a quadratic trinomial in P. At the circle (5.1) we have

$$I(\alpha_1, P) = a(\alpha_1)P^2 + b(\alpha_1)P + c(\alpha_1) \tag{5.2}$$

The coefficients of the trinomial (5.2) were computed for $\theta_0^2 = 9.7157$ by the author. The computations have shown that the trinomial has no positive roots no matter what values are assumed by α_1, and this implies that the theorem is applicable for any $(\alpha, \beta) \in E$. We note that the quantity I also plays a part in investigating the stability of the resulting secondary modes relative to the perturbations of the same periodicity.

Note. All previous arguments are extended to the case of convection in a layer between two free surfaces. In this case the sign of the functional I can be determined analytically. It can be shown that in this case

$$2I = -(D(\Phi_{12} + \Phi_{21}),\ \Phi_{12} + \Phi_{21})_H < 0$$

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